Available online at www.sciencedirect.com

ScienceDirect

Journal of Computational and Applied Mathematics 208 (2006) 111–119

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Eigenvalues of the radial p-Laplacian with a potential on $(0, \infty)$

B.M. Brown*, M.S.P. Eastham

Department of Computer Science, University of Cardiff, Cardiff CF24 3XF, UK

Received 18 January 2006

Abstract

Brown and Reichel recently established the existence of eigenvalues for the p-Laplacian on $\mathbf{R}^+$ when the potential q is either (i) large and positive or (ii) sufficiently large and negative ("limit-circle" case) at infinity. Their methods imposed extra restrictions on q. In this paper, these restrictions are removed. In addition, the case where q decays at infinity is also shown to produce negative eigenvalues, and a condition is given under which there are only a finite number of such eigenvalues.

MSC: 34A34

Keywords: Periodic eigenvalues; Prüfer transformation; P-Laplacian; Rotation number

1. Introduction

In a recent paper [2], Brown and Reichel initiated a spectral theory for the radial p-Laplacian on $(0, \infty)$ in the two cases where (i) the potential $q(r) \to \infty$ as $r \to \infty$ and (ii) $q(r) = -r^\alpha$ with $\alpha > p/(p-1)$. They established the existence of a countable infinity of eigenvalues in analogy with the classical case $p = 2$. The method developed in [2] is a new one based on Prüfer-type transformations which involve p-generalisations of the sine and cosine functions. It does however also involve additional growth and differentiability conditions on q in case (i), while case (ii) is more or less confined to $-q$ being a power of r. In [2, Section 5], the question was raised whether these restrictions can be lifted. Our purpose in this paper is to show that these restrictions can indeed be dispensed with and also to deal similarly with the situation where $q(r) \to 0$ as $r \to \infty$.

As explained in [2], the eigenvalue problem arising from the spherically symmetric p-Laplacian in $\mathbf{R}^\mathbf{n}$ is

$$-(r^{n-1}u'^{(p-1)})' + r^{n-1}q(r)u^{(p-1)} = \lambda r^{n-1}u^{(p-1)} \tag{1.1}$$

on $(0, \infty)$ with the boundary condition

$$u'(0) = 0. \tag{1.2}$$

Here $p > 1$ and, with f as either u or u', the power notation is

$$f^{(p-1)} = |f|^{p-2}f \tag{1.3}$$

* Corresponding author.
E-mail address: malcolm.brown@cs.cardiff.ac.uk (B.M. Brown).

0377-0427/$ - see front matter © 2006 Published by Elsevier B.V.
doi:10.1016/j.cam.2006.10.046

[7, p. 175]. The potential q is continuous on $[0, \infty)$, and both u and $u'^{(p-1)}$ are $\mathbf{C}^1[0, \infty)$. Finally, λ is said to be a eigenvalue if u is non-trivial and

$$u \in L^p(0, \infty; r^{n-1}). \tag{1.4}$$

We require the same p-generalisation $S_p(r)$ of the sine function which was discussed in [2, Section 2.1]. Briefly, S_p is the solution of

$$(s')^p + s^p/(p-1) = 1$$

in $[0, \pi_p/2]$ such that $s(0) = 0$ and $s'(0) = 1$, where

$$\pi_p = 2\pi(p-1)^{1/p}/\{p \sin(\pi/p)\}.$$

Then

$$S_p(r) = \begin{cases} S_p(\pi_p - r), & \pi_p/2 \leqslant r \leqslant \pi_p, \\ -S_p(2\pi_p - r), & \pi_p \leqslant r \leqslant 2\pi_p. \end{cases}$$

Outside $[0, 2\pi_p]$, S_p is defined by $2\pi_p$-periodicity. Thus S_p and S_p' have the same signs as $\sin x$ and $\cos x$ in the corresponding parts of $(0, 2\pi)$. Also, for all r,

$$|S_p'|^p + |S_p|^p/(p-1) = 1 \tag{1.5}$$

and

$$(S_p'^{(p-1)})' + S_p^{(p-1)} = 0. \tag{1.6}$$

A full account of S_p is given in [5].

2. The case $q \to \infty$

This case was discussed in [2, Section 3] using the generalised Prüfer transformation

$$u^{(p-1)} = \rho\{S_p(\phi)\}^{(p-1)}, \tag{2.1}$$

$$r^{n-1}(u')^{(p-1)} = \rho\{S_p'(\phi)\}^{(p-1)}, \tag{2.2}$$

which leads to the first-order differential equations for ϕ and ρ

$$\phi' = \frac{r^{n-1}}{p-1}(-q+\lambda)|S_p(\phi)|^p + r^{(1-n)/(p-1)}|S_p'(\phi)|^p, \tag{2.3}$$

$$\rho'/\rho = \{r^{n-1}(q-\lambda) + r^{(1-n)/(p-1)}\}\{S_p(\phi)\}^{(p-1)}S_p'(\phi). \tag{2.4}$$

Note however that the method of proof in [2] depends on the construction of quite intricate sub- and super-functions for (2.3) which entail supplementary conditions on q of the type

$$q(r) \geqslant \alpha r^\beta, \quad q'/q^{1+1/p} = \mathrm{o}(1) \ \ (r \to \infty). \tag{2.5}$$

Here we prove the same theorem as [2, Theorem 1] but without these conditions.

Theorem 2.1. *Let $q(r) \to \infty$ as $r \to \infty$. Then the problem* (1.1)–(1.2) *has a countable number of simple eigenvalues $\lambda_0 < \lambda_1 < \cdots$ with $\lambda_k \to \infty$ as $k \to \infty$, and no other eigenvalues. The corresponding eigenfunction u_k has k simple zeros in* $(0, \infty)$.

We give the proof after the following four lemmas which are basically in [2], but we wish to make it clear that they do not require (2.5). As in [2, Remark 7], we consider only *tamed solutions* $\phi(r)$ of (2.3), meaning that $\phi(r) = \pi_p/2 + \mathrm{O}(r^n)$

or, more precisely,

$$\phi(r) = \pi_p/2 + \left\{\frac{1}{n}(-q(0) + \lambda) + \mathrm{o}(1)\right\} r^n \tag{2.6}$$

as $r \to 0$ [1, Lemma 9], [2, Lemma 6].

Lemma 2.2. *Let $q(r) \to \infty$ as $r \to \infty$. Then*

$$\phi(r) \to N\pi_p \tag{2.7}$$

as $r \to \infty$, where N is a non-negative integer depending on λ.

Proof. Let $[R, \infty)$ be an interval in which $q > \lambda$. Then, as noted in [2, Lemma 10], the constant function

$$\psi(r) = (M + \tfrac{1}{2})\pi_p$$

is a superfunction for (2.3) in $[R, \infty)$ for a suitable integer M. Also, the zero function is a subfunction in $[0, \infty)$. Hence $0 < \phi(r) < (M + \frac{1}{2})\pi_p$ in $[R, \infty)$, and so

$$L = \limsup \phi(r), \quad l = \liminf \phi(r) \ \ (r \to \infty)$$

are both finite. We have to show that $L = l = N\pi_p$.

We recall that, as usual with Prüfer equations such as (2.3), ϕ cannot decrease through a value $m\pi_p$ (m an integer) because $\phi' = r^{(1-n)}/(p-1) > 0$ at such a point. Similarly, in $[R, \infty)$, ϕ cannot increase through a value $(m + \frac{1}{2})\pi_p$ because

$$\phi' = r^{n-1}(-q + \lambda) < 0$$

at such a point. It follows that

$$0 \leqslant L - l \leqslant \pi_p/2. \tag{2.8}$$

If $L > l$, we can let $r \to \infty$ in (2.3) through a sequence of values where $\phi' = 0$. This gives $S_p(L) = S_p(l) = 0$. Hence, by (2.8), we have the contradiction $L = l$. Thus the only possibility in (2.8) is $L = l$ with $S_p(L) = 0$, proving (2.7).

For each N, there are two cases of (2.7) to consider, depending on λ. Since ϕ cannot decrease through the value $N\pi_p$, (2.7) means that, in a neighbourhood of ∞, either

(a)

$$\phi(r) > N\pi_p \tag{2.9}$$

or

(b)

$$\phi(r) < N\pi_p. \qquad \square \tag{2.10}$$

In the next lemma, we consider case (a) and we indicate the dependence of ϕ on both r and λ.

Lemma 2.3. *Let Λ_N denote the set of values of λ for which* (2.7) *holds with a given N, and $\phi(r, \lambda) > N\pi_p$ in a neighbourhood of ∞. Then Λ_N is an open interval.*

Proof. Let $\lambda \in \Lambda_N$ and choose R so that $q > \lambda$ in $[R, \infty)$ and so that $N\pi_p < \phi(R, \lambda) < (N + \frac{1}{2})\pi_p$. Then, for all λ' sufficiently close to λ, $\phi(R, \lambda')$ satisfies the same inequalities, and $q > \lambda'$. Hence, as noted above, $\phi(r, \lambda')$ cannot decrease through the value $N\pi_p$ nor increase through the value $(N + \frac{1}{2})\pi_p$. By Lemma 2.2, the only possibility for

$\phi(r, \lambda')$ is that $\phi(r, \lambda') \to N\pi_p$ from above. Hence $\lambda' \in \Lambda_N$ as required. The conclusion that Λ_N is an interval follows from the monotonicity of $\phi(r, \lambda)$ as a function of λ. $\square$

Now we move on to case (b).

Lemma 2.4. *For a given N, there is at most one value of λ for which (2.7) holds and $\phi(r, \lambda) < N\pi_p$ in a neighbourhood of ∞.*

The proof of this result is given in [2, Proposition 14] without the need (we note) for the extra conditions (2.5).

Lemma 2.5. *The open interval Λ_0 has the form $(-\infty, \lambda_0)$.*

Proof. Consider any $\lambda < \min q(r)$ $(0 \leqslant r < \infty)$. Then, by (2.6), $\phi(r, \lambda) < \pi_p/2$ in some interval $(0, r_0)$. Further, as we have noted already, the fact that $q > \lambda$ for all r means that $\phi(r, \lambda)$ cannot increase through the value $\pi_p/2$ in $(0, \infty)$. Thus the only possibility in (2.7) is $N = 0$ with (2.9) holding since always $\phi(r, \lambda) > 0$ in $(0, \infty)$. $\square$

Proof of Theorem 2.1. The intervals Λ_k $(k = 0, 1, 2, \ldots)$, being open and disjoint, cannot cover the whole of the λ-range $(-\infty, \infty)$. The complement of the intervals can only consist of isolated points, by Lemma 2.4. Thus the Λ_k have the form $(\lambda_{k-1}, \lambda_k)$ with $\lambda_{-1} = -\infty$ by Lemma 2.5. We note that the solution u_k corresponding to λ_k has precisely k zeros in $(0, \infty)$, by (2.10) with $N = k + 1$.

Now we show that u_k is an eigenfunction, satisfying (1.4). The argument is similar to that used in [6, p. 110]. By (2.10) we have $\phi(r, \lambda_k) < (k + 1)\pi_p$ in a neighbourhood of ∞, and hence

$$S_p(\phi) > 0, \quad S_p'(\phi) < 0 \tag{2.11}$$

when k is even. (When k is odd, these inequalities are reversed and similar reasoning to what follows is applicable.) Recalling the definition (1.3), we deduce from (2.1), (2.2) and (2.11) that

$$u_k > 0, \quad u_k' < 0$$

near ∞. Hence

$$u_k(r) \to l \quad (\geqslant 0) \tag{2.12}$$

as $r \to \infty$. Next, again for r_1 and r_2 suitably large, integration of (1.1) gives

$$\begin{aligned}\int_{r_1}^{r_2} r^{n-1}(q - \lambda_k)u_k^{p-1}\,\mathrm{d}r &= [r^{n-1}u_k'^{\,(p-1)}]_{r_1}^{r_2} \\ &= -[r^{n-1}|u_k'|^{p-1}]_{r_1}^{r_2} < r_1^{n-1}|u_k'(r_1)|^{p-1}.\end{aligned}$$

Then, letting $r_2 \to \infty$, we have

$$r^{n-1}(q - \lambda_k)u_k^{p-1} \in L(r_1, \infty) \tag{2.13}$$

and a fortiori $|u_k|^{p-1} \in L(0, \infty)$. It follows that $l = 0$ in (2.12) and then, in turn, it follows from (2.13) that (1.4) holds for u_k as required.

To complete the proof of the theorem, we have to show that u does not satisfy (1.4) when λ is not any of the λ_k. In this situation (2.9) holds and, considering for example N to be odd, we have

$$S_p(\phi) < 0, \quad S_p'(\phi) < 0$$

in place of (2.11). It then follows from (1.3), (2.1), and (2.2) that

$$u < 0, \quad u' < 0$$

near ∞. Hence certainly (1.4) cannot hold and therefore λ is not an eigenvalue. This completes the proof of Theorem 2.1. $\square$

3. The case $q \to 0$

There is a corresponding result to Theorem 2.1 which again generalises the classical $p=2$ situation when $q(r) \to 0$ as $r \to \infty$.

Theorem 3.1. *Let $q(r) \to 0$ as $r \to \infty$. Then problem* (1.1)–(1.2) *has a finite or countably infinite number of negative eigenvalues $\lambda_0 < \lambda_1 < \cdots < 0$, and no other negative eigenvalues. The corresponding eigenfunction u_k has k simple zeros in* $(0, \infty)$.

Proof. We are now considering $\lambda < 0$ and therefore, as above, we have $q - \lambda > 0$ in some interval $[R, \infty)$. Hence we have the same conclusion that ϕ cannot increase through a value $(m + \frac{1}{2})\pi_p$. In addition, when $n > 1$, the second term on the right of (2.3) is negligible compared to the first when $r \to \infty$, and we can therefore argue precisely as before to prove Theorem 3.1 when $n > 1$. When $n = 1$, however, the two terms on the right of (2.3) are of comparable size, and we have now to re-examine our use of (2.3).

Considering now $n = 1$, we can use (1.5) to write (2.3) as

$$\phi' = 1 - \frac{1}{p-1}(1 + q - \lambda)|S_p(\phi)|^p. \tag{3.1}$$

As in the proof of Lemma 2.2, we can prove again that $\phi(r) \to L$, a finite limit, as $r \to \infty$ and, by (3.1),

$$|S_p(L)| = \left(\frac{p-1}{1-\lambda}\right)^{1/p}. \tag{3.2}$$

Then also (1.5) gives

$$|S_p'(L)| = \left(\frac{-\lambda}{1-\lambda}\right)^{1/p}. \tag{3.3}$$

There are now two possibilities, in analogy with (2.9) and (2.10).

(a)

$$k\pi_p < L < (k + \tfrac{1}{2})\pi_p, \tag{3.4}$$

(b)

$$(k + \tfrac{1}{2})\pi_p < L < (k+1)\pi_p. \tag{3.5}$$

Here k is a non-negative integer. In case (a), $S_p(\phi)$ and $S_p'(\phi)$ have the same sign near to ∞, in case (b) opposite signs.

Consider now the set Λ_k of values of λ for which (3.4) holds with a given k. Then, much as in the proof of Lemma 2.3, it follows again that Λ_k is an open interval. We do however give a word of proof that there is a unique value of λ for which (3.5) holds.

Suppose then that there are two such values λ_1 and λ_2 with $\lambda_1 < \lambda_2$. By (3.1), $\phi(r, \lambda_1) < \phi(r, \lambda_2)$ as usual. If, for example, k is even in (3.5), we then have

$$S_p(\phi(r, \lambda_1)) > S_p(\phi(r, \lambda_2)) > 0$$

near ∞. Letting $r \to \infty$ and using (3.2), we obtain

$$(p-1)/(1-\lambda_1) \geqslant (p-1)/(1-\lambda_2),$$

giving the contradiction $\lambda_1 \geqslant \lambda_2$; similarly if k is odd.

The proof of Theorem 3.1 now continues to a conclusion in the same way as Theorem 2.1. $\square$

In the next theorem we give a condition on q which guarantees that there are only a finite number of negative eigenvalues. This condition is a direct generalisation of a familiar non-oscillation criterion when $p=2$ [4, p. 96].

Theorem 3.2. *Let $p>n$ and let*

$$q(r) \geqslant -kr^{-p} \tag{3.6}$$

in some interval $[r_1, \infty)$, where

$$k = \{(p-n)/p\}^p. \tag{3.7}$$

Then there are only finitely many negative eigenvalues.

Proof. We continue to use the solution $\phi(r, \lambda)$ of (2.3) and (2.6), and we note that, as usual, (2.3) gives

$$\phi(r, \lambda) < \phi(r, 0) \tag{3.8}$$

in $(0, \infty)$ for $\lambda < 0$. Now consider the example

$$-(r^{n-1}u'^{(p-1)})' - kr^{n-1-p}u^{(p-1)} = 0 \quad (r_1 \leqslant r < \infty) \tag{3.9}$$

of (1.1) with the solution

$$u(r) = r^{1-n/p}. \tag{3.10}$$

The Prüfer equation (2.3) corresponding to (3.9) and (3.10) is

$$\psi' = \{k/(p-1)\}r^{n-1-p}|S_p(\psi)|^p + r^{(1-n)/(p-1)}|S_p'(\psi)|^p. \tag{3.11}$$

The initial condition at r_1 given by (2.1), (2.2) and (3.10) is

$$\tan_p \psi(r_1) := (S_p/S_p')(\psi(r_1)) = \left(1 - \frac{n}{p}\right)^{-1} r_1^{(p-n)/(p-1)}$$

with

$$0 < \psi(r_1) < \pi_p/2.$$

Since u and u' have no zeros in $[r_1, \infty)$, we have

$$0 < \psi(r) < \pi_p/2 \quad (r_1 \leqslant r < \infty). \tag{3.12}$$

Now define

$$\psi_1 = \psi + M\pi_p, \tag{3.13}$$

where the integer M is chosen so that

$$\psi(r_1) + M\pi_p > \phi(r_1, 0).$$

Then ψ_1 also satisfies (3.11) and, by (3.6) and (2.3), $\phi(r, 0)$ is a subfunction to ψ_1 in $[r_1, \infty)$. Hence

$$\phi(r, 0) < \psi_1(r) < (M + \tfrac{1}{2})\pi_p$$

by (3.12) and (3.13). Hence again, by (3.8), ϕ is bounded independently of λ, thus precluding an infinity of eigenvalues as characterised by Theorem 3.1. $\square$

4. The case $q \to -\infty$

We turn now to the second main situation considered in [2] where $q(r) \to -\infty$ as $r \to \infty$, and the question is whether all solutions of (1.1) satisfy (1.4). In the following theorem, we answer the open question (2) in [2, Section 5] by allowing a broader class of potentials much as in the classical $p=2$ case. We do this by choosing a more appropriate Prüfer transformation than the one in [2, Section 4, (2.5) and (2.6)]. This choice is governed by the simple formula

$$|S_p'|^p - |S_p|^p = (S_p S_p'^{(p-1)})' \tag{4.1}$$

which follows from (1.6) and (1.3) when the product on the right-hand side is differentiated. When $p=2$, (4.1) reduces to $\cos^2 r - \sin^2 r = \cos 2r$.

Theorem 4.1. *Let $q(r) \to -\infty$ as $r \to \infty$. Also, for some $r_1 > 0$, let q' be $AC_{\mathrm{loc}}(r_1, \infty)$ and let*

$$q'^2/q^{2+1/p} \in L(r_1, \infty), \quad q''/q^{1+1/p} \in L(r_1, \infty). \tag{4.2}$$

Finally, let

$$q^{-(p-1)/p} \in L(r_1, \infty). \tag{4.3}$$

Then all solutions of (1.1) *satisfy* (1.4).

Proof. For any given λ we work in an interval (r_1, ∞) where $q(r) < \lambda$, and we introduce the Prüfer transformation

$$u^{(p-1)} = \rho Q^\alpha r^\gamma \{S_p(\phi)\}^{(p-1)}, \tag{4.4}$$

$$r^{n-1}(u')^{(p-1)} = \rho Q^\beta r^\delta \{S_p'(\phi)\}^{(p-1)} \tag{4.5}$$

in which $Q = \lambda - q$ (>0) and the constants α, β, γ and δ are to be chosen. This is of course a generalisation of (2.1)–(2.2), and we note that the (inefficient) choice in [2, Section 4] is $\alpha = -(p-1)/p, \gamma = -(n-1), \beta = \delta = 0$. Now (4.4) and (4.5) lead to first-order differential equations for ϕ and ρ which correspond to (2.3) and (2.4), and to [2, (9) and (10)]. We write

$$A = n - 1 + \gamma - \delta. \tag{4.6}$$

Then, omitting the details of the calculations, we obtain

$$\begin{aligned} \phi' &= Q^{(\beta-\alpha)/(p-1)} r^{-A/(p-1)} + \frac{1}{p-1}\{Q^{\alpha-\beta+1} r^A - Q^{(\beta-\alpha)/(p-1)} r^{-A/(p-1)}\}|S_p|^p \\ &\quad - \frac{1}{p-1}\left\{(\alpha-\beta)\frac{Q'}{Q} + (\gamma-\delta)\frac{1}{r}\right\} S_p S_p'^{(p-1)}, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \frac{\rho'}{\rho} &= \{Q^{(\beta-\alpha)/(p-1)} r^{-A/(p-1)} - Q^{\alpha-\beta+1} r^A\} S_p^{(p-1)} S_p' \\ &\quad - \left\{\frac{\alpha}{p-1}|S_p|^p + \beta |S_p'|^p\right\}\frac{Q'}{Q} - \left\{\frac{\gamma}{p-1}|S_p|^p + \delta|S_p'|^p\right\}\frac{1}{r}. \end{aligned} \tag{4.8}$$

We can now move on to our choice of α, β, γ and δ. First, we arrange that the coefficient of $|S_p|^p$ in (4.7) is zero. Then $\beta - \alpha = (p-1)/p$ and $A = 0$, i.e. $\delta - \gamma = n - 1$ by (4.6) Then, in (4.8), we arrange for the appearance of (4.1) in the coefficients of Q'/Q and $1/r$. Thus we require $\alpha/(p-1) = -\beta$ and $\gamma/(p-1) = -\delta$. Hence, altogether, the choice is

$$\alpha = -(p-1)^2/p^2, \quad \beta = (p-1)/p^2, \quad \gamma = -(n-1)(p-1)/p, \quad \delta = (n-1)/p. \tag{4.9}$$

Now (4.7) and (4.8) simplify to

$$\phi' = Q^{1/p} + \frac{F'}{F}\sigma, \tag{4.10}$$

$$\frac{\rho'}{\rho} = -\frac{p-1}{p}\frac{F'}{F}\sigma', \tag{4.11}$$

where

$$F = Q^{1/p} r^{(n-1)/(p-1)} \tag{4.12}$$

and

$$\sigma = S_p S_p'^{(p-1)}. \tag{4.13}$$

We aim to show that ρ is bounded as $r \to \infty$, and we denote by $\mathscr{L}$ any term which is $L(r_1, \infty)$ as a result of (4.2). We also recall that, since S_p is a function of ϕ, the primes on terms involving S_p in (4.10) and (4.11) denote differentiation with respect to ϕ. Thus, by (4.10), we can write (4.11) as

$$\frac{\rho'}{\rho} = -\frac{p-1}{p} Q^{-1/p}\frac{F'}{F}\sigma'\left(\phi' - \frac{F'}{F}\sigma\right).$$

Now S_p and its derivatives are bounded [2, Lemma 5], and it then follows from (4.2), (4.12) and (4.13) that

$$\frac{\rho'}{\rho} = -\frac{p-1}{p} Q^{-1/p}\frac{F'}{F}\sigma'\phi' + \mathscr{L}.$$

Integrating over (r_1, r) and integrating by parts on the right, we obtain

$$\begin{aligned}\log\rho &= \frac{p-1}{p}\int_{r_1}^{r}\left(Q^{-1/p}\frac{F'}{F}\right)'\sigma\,\mathrm{d}t + (\text{const.}) + \mathrm{o}(1)\\ &= (\text{const.}) + \mathrm{o}(1)\end{aligned} \tag{4.14}$$

as $r \to \infty$ by (4.2) again. It now follows from (4.4) and (4.9) that

$$|u|^{p-1} \leqslant (\text{const.})|q|^{-(p-1)^2/p^2} r^{-(n-1)(p-1)/p}$$

and hence

$$|u|^p r^{n-1} \leqslant (\text{const.})|q|^{-(p-1)/p}.$$

Hence, finally, (4.3) shows that (1.4) is satisfied by all solutions u.

In the case $p=2$, Theorem 4.1 becomes the standard limit-circle result which is obtained from the Liouville–Green asymptotic formulae [3, Corollary 2.2.1 and Theorem 2.5.1] and the simplest example is $q(r) = -r^c \quad (c>2)$. With a little more work we can also obtain the complementary " limit-point " result where (4.3) does not hold and $0 < c \leqslant 2$. This situation is not covered in [2]. □

Theorem 4.2. *Let* (4.2) *hold with, in addition,* $q' \leqslant 0$, *and let*

$$q^{-(p-1)/p} \notin L(r_1, \infty). \tag{4.15}$$

Then no non-trivial solutions of (1.1) *satisfies* (1.4).

Proof. From (4.4) and (4.9) we have

$$|u|^p = \rho^{p/(p-1)} Q^{-(p-1)/p} r^{-(n-1)} |S_p|^p$$

and hence, by (1.5) and (4.1),

$$r^{n-1}|u|^p = \{(p-1)/p\}(\rho^{p/(p-1)} Q^{-(p-1)/p} - V),$$

where

$$V = \rho^{p/(p-1)} Q^{-(p-1)/p} \sigma'. \tag{4.16}$$

By (4.14) and (4.15), the theorem will be proved if we show that $\int^{\infty} V \, \mathrm{d}r$ converges.

By (4.10) and (4.11), we write (4.16) as

$$V = \frac{\mathrm{d}}{\mathrm{d}r}(\rho^{p/(p-1)} Q^{-1} \sigma) + \rho^{p/(p-1)} Q' Q^{-2} \sigma.$$

Now the second term on the right is $L(r_1, \infty)$ because $Q'(= -q') > 0$ and ρ and σ are bounded. Thus, $\int^{\infty} V \, \mathrm{d}r$ converges as required. $\square$

References

[1] B.M. Brown, W. Reichel, Computing eigenvalues and Fučík-spectrum of the radially symmetric p-Laplacian, J. Comput. Appl. Math. 148 (1) (2002) 183–211.
[2] B.M. Brown, W. Reichel, Eigenvalues of the radially symmetric p-Laplacian in $\mathbf{R}^{\mathbf{n}}$, J. London Math. Soc. 69 (2–3) (2004) 657–675.
[3] M.S.P. Eastham, The asymptotic solution of linear differential systems, London Mathematical Society Monographs, vol. 4, Clarendon Press, Oxford, 1989.
[4] I.M. Glazman, Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators, Israel Program for Scientific Translations Ltd., Jerusalem, 1965.
[5] P. Lindqvist, Some remarkable sine and cosine functions, Ricerche Mat. 44 (1995) 269–290.
[6] E.C. Titchmarsh, Eigenfunction Expansions Associated with Second-order Differential Equations, Part I, second ed., Clarendon Press, Oxford, 1962.
[7] W. Walter, Sturm–Liouville theory for the radial Δ_p-operator, Math. Z. 227 (1) (1998) 175–185.